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Estimates for eigenvalues of the Paneitz operator ☆

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Abstract

For an n-dimensional compact submanifold M^n in the Euclidean space $\mathbf{R}^N$, we study estimates for eigenvalues of the Paneitz operator on M^n. Our estimates for eigenvalues are sharp.
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1. Introduction

For compact Riemann surfaces M^2, Li and Yau [11] introduced the notion of conformal volume, which is a global invariant of the conformal structure. They determined the conformal volume for a large class of Riemann surfaces, which admit minimal immersions into spheres. In particular, they proved that for a compact Riemann surface M^2, if there exists a conformal map from M^2 into the unit sphere $S^N(1)$, then the first eigenvalue λ_1 of the Laplacian satisfies

$$\lambda_1 \operatorname{vol}\bigl(M^2\bigr) \le 2V_c\bigl(N, M^2\bigr)$$

and the equality holds only if M^2 is a minimal surface in $S^N(1)$, where $V_c(N, M^2)$ is the conformal volume of M^2.

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For 4-dimensional compact Riemannian manifolds, Paneitz [13] introduced a fourth order operator P_g defined by, letting div be the divergence for the metric g,

$$P_g f = \Delta^2 f - \operatorname{div}\left[\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)\nabla f\right], \tag{1.1}$$

for smooth functions f on M^4, where Δ and ∇ denote the Laplacian and the gradient operator with respect to the metric g on M^4, respectively, and R and Ric are the scalar curvature and Ricci curvature tensor with respect to the metric g on M^4. Furthermore, Branson [1] has generalized the Paneitz operator to an n-dimensional Riemannian manifold. For an n-dimensional Riemannian manifold (M^n, g), the operator P_g is defined by

$$P_g f = \Delta^2 f - \operatorname{div}\big[(a_n Rg + b_n \mathrm{Ric})\nabla f\big] + \frac{n-4}{2} Qf, \tag{1.2}$$

where

$$Q = c_n|\mathrm{Ric}|^2 + d_n R^2 - \frac{1}{2(n-1)}\Delta R$$

is called Q-curvature with respect to the metric g,

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}, \qquad b_n = -\frac{4}{n-2},$$
$$c_n = -\frac{2}{(n-2)^2}, \qquad d_n = \frac{n(n-2)^2-16}{8(n-1)^2(n-2)^2}.$$

This operator P_g is also called Paneitz operator or Branson–Paneitz operator. It is known that Paneitz operator is conformally invariant of bi-degree $(\frac{n-4}{2}, \frac{n+4}{2})$, that is, under conformal transformation of Riemannian metric $g = e^{2w} g_0$, the Paneitz operator P_g changes into

$$P_g f = e^{-\frac{n+4}{2}w} P_{g_0}\big(e^{\frac{n-4}{2}w} f\big). \tag{1.3}$$

Let $\mathfrak{M}(M^n)$ be the set of Riemannian metrics on M^n. For each $g \in \mathfrak{M}(M^n)$, the total Q-curvature for g is defined by

$$Q[g] = \int_{M^n} Q\,dv.$$

When $n = 4$, from the Gauss–Bonnet–Chern theorem for dimension 4, we have

$$Q[g] = -\frac{1}{4}\int_{M^4} |W|^2 dv + 8\pi^2 \chi\big(M^4\big), \tag{1.4}$$

where W is the Weyl conformal curvature tensor and $\chi(M^4)$ is the Euler characteristic of M^4. Hence, we know that the total Q-curvature is a conformal invariant for dimension 4.

We should remark that Paneitz operator and Q-curvature for 4-dimensional manifolds M^4 enjoy similar conformal property under conformal change of metric as the Laplacian and Gaussian curvature do in dimension 2.

In [12], Nishikawa has studied the variation of the total Q-curvature for a general dimension n. He has proved that a Riemannian metric g on an $n(n \neq 4)$-dimensional compact manifold M^n is a critical point of the total Q-curvature functional with respect to a volume preserving conformal variation of the metric g, if and only if the Q-curvature with respect to the metric g is constant.

Furthermore, an important problem in conformal geometry is to construct conformal metrics for which a certain curvature quantity equals to a prescribed function (for examples, a constant). Brendle [3] has proved for a 4-dimensional compact Riemannian manifold M^4, if the Paneitz operator P_g has non-negative eigenvalues, its kernel consists of the constant functions and the total Q-curvature satisfying

$$Q[g] < 8\pi^2$$

then there exists a conformal metric $\tilde{g}$ of g on M^4 such that the Q-curvature with respect to the metric $\tilde{g}$ is a constant multiple of a prescribed positive function. Hence, it is very important to study eigenvalues of the Paneitz operator P_g.

Gursky [9] have studied eigenvalues of the Paneitz operator for $n = 4$. Since the Paneitz operator P_g is an elliptic operator and $P_g 1 = 0$ for $n = 4$, we know that $\lambda_0 = 0$ is an eigenvalue of P_g. Gursky [9] shown that if the Yamabe invariant of M^4 is non-negative and the total Q-curvature is non-negative, the first eigenvalue λ_1 is positive. For $n \geq 6$, Yang and Xu [14] have proved the Paneitz operator P_g is positive if the scalar curvature is positive and Q-curvature is nonnegative. Furthermore, see [2,4,5,10].

For $n \geq 3$, we consider the following closed eigenvalue problem on an n-dimensional compact manifold M^n:

$$P_g u = \lambda u. \tag{1.5}$$

Since P_g is an elliptic operator, the spectrum of P_g on M^n is discrete. We assume

$$0 < \lambda_1 < \lambda_2 \leq \cdots, \lambda_k \leq \cdots \to +\infty$$

for $n \neq 4$ and for $n = 4$,

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots, \lambda_k \leq \cdots \to +\infty.$$

When $n = 4$, Yang and Xu [15] have introduced an N-conformal energy $E_c(N, M^4)$ if M^4 can be conformally immersed into the unit sphere $S^N(1)$ and have obtained an upper bound for the first eigenvalue λ_1:

$$\lambda_1 \operatorname{vol}\bigl(M^4\bigr) \leq E_c\bigl(N, M^4\bigr),$$

where $\operatorname{vol}(M^n)$ denotes the volume of M^n. Furthermore, Chen and Li [7] have also studied the upper bound on the first eigenvalue λ_1 when M^4 is considered as a compact submanifold in

a Euclidean space $\mathbf{R}^N$. They have proved

$$\lambda_1 \leq \frac{\int_{M^4}(16|H|^2 + \frac{2}{3}R)dv \int_{M^4}|H|^2 dv}{\{\mathrm{vol}(M^4)\}^2}$$

and the equality holds if and only if M^4 is a minimal submanifold in a sphere $S^{N-1}(r)$ for $N > 5$ and M^4 is a round sphere $S^4(r)$ for $N = 5$. In [8], the second eigenvalue λ_2 of the Paneitz operator P_g is studied. By making use of the conformal transformation introduced by Li and Yau [11], Chen and Li proved, for $n \geq 7$,

$$\lambda_2 \,\mathrm{vol}\big(M^n\big) \leq \frac{1}{2}n\big(n^2 - 4\big)\int_{M^n}|H|^4 dv + \frac{n-4}{2}\int_{M^n} Q dv$$

if M^n is a compact submanifold in the Euclidean space $\mathbf{R}^N$. Here $|H|$ denotes the mean curvature of M^n in $\mathbf{R}^N$. As they remarked, their method does not work for $3 \leq n \leq 6$.

The purpose of this paper is to study eigenvalues of the Paneitz operator P_g in n-dimensional compact Riemannian manifolds. Our method is very different from one used by Chen and Li [8] and Xu and Yang [15]. From Nash's theorem, we know that each compact Riemannian manifold can be isometrically immersed into a Euclidean space $\mathbf{R}^N$. Thus, we can assume M^n is an n-dimensional compact submanifold in $\mathbf{R}^N$.

Theorem 1.1. *Let (M^4, g) be a 4-dimensional compact submanifold with the metric g in $\mathbf{R}^N$. Then, eigenvalues of the Paneitz operator P_g satisfy*

$$\sum_{j=1}^{4}\lambda_j^{\frac{1}{2}} \leq 4\frac{\sqrt{\int_{M^4}(16|H|^2 + \frac{2}{3}R)dv \int_{M^4}|H|^2 dv}}{\mathrm{vol}(M^4)}$$

and the equality holds if and only if M^4 is a round sphere $S^4(r)$ for $N = 5$ and M^4 is a compact minimal submanifold with constant scalar curvature in $S^{N-1}(r)$ for $N > 5$.

Corollary 1.1. *Let (M^4, g) be a 4-dimensional compact submanifold with the metric g in the unit sphere $S^N(1)$. Then, eigenvalues of the Paneitz operator P_g satisfy*

$$\sum_{j=1}^{4}\lambda_j^{\frac{1}{2}} \leq 4\frac{\sqrt{\int_{M^4}(16|H|^2 + 16 + \frac{2}{3}R)dv \int_{M^4}(|H|^2 + 1)dv}}{\mathrm{vol}(M^4)}$$

and the equality holds if and only if M^4 is a compact minimal submanifold with constant scalar curvature in $S^N(1)$.

Theorem 1.2. *Let (M^n, g) $(n > 4)$ be an n-dimensional compact submanifold with the metric g in $\mathbf{R}^N$. Then, eigenvalues of the Paneitz operator P_g satisfy*

$$\begin{aligned}\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)^{\frac{1}{2}} &\\ \leq \sqrt{\int_{M^n}\frac{n(n^2-4)|H|^2}{2}u_1^2dv+2(n+2)\int_{M^n}g(\nabla u_1,\nabla u_1)dv} &\\ \times\sqrt{\int_{M^n}n^2|H|^2u_1^2dv+4\int_{M^n}g(\nabla u_1,\nabla u_1)dv}&\end{aligned}$$

and the equality holds if and only if M^n is isometric to a sphere $S^n(r)$, where u_1 is the normalized first eigenfunction of P_g.

Remark 1.1. In our Theorem 1.2, we do not need to assume the positivity of the Paneitz operator P_g.

If the Paneitz operator P_g is a positive operator, we have

Theorem 1.3. *Let (M^n, g) $(n \neq 4)$ be an n-dimensional compact submanifold with the metric g in the unit sphere $S^N(1)$. Then, eigenvalues of the Paneitz operator P_g satisfy*

$$\sum_{j=1}^{n}\lambda_j^{\frac{1}{2}} < n\frac{\sqrt{\int_{M^n}((n^2|H|^2+n^2)+(na_n+b_n)R+\frac{n-4}{2}Q)dv\int_{M^n}(|H|^2+1)dv}}{\mathrm{vol}(M^n)}.$$

2. Eigenvalues of the Paneitz operator on M^4

Assume that M^n is an n-dimensional submanifold in $\mathbf{R}^N$. Let $(x_1, \cdots, x_n)$ be a local coordinate system in a neighborhood U of $p \in M^n$. Let $\mathbf{y}$ be the position vector of p in $\mathbf{R}^N$, which is defined by

$$\mathbf{y} = \big(y_1(x_1, \cdots, x_n), \cdots, y_N(x_1, \cdots, x_n)\big).$$

Let g denote the induced metric of M^n from $\mathbf{R}^N$ and $\langle,\rangle$ is the standard inner product in $\mathbf{R}^N$. Chen and Cheng [6] have proved the following:

Lemma 2.1. *For any function $u \in C^\infty(M^n)$, we have*

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left\langle\sum_{\alpha=1}^{N}\frac{\partial y_\alpha}{\partial x_i}\frac{\partial}{\partial y_\alpha}, \sum_{\beta=1}^{N}\frac{\partial y_\beta}{\partial x^i}\frac{\partial}{\partial y_\beta}\right\rangle = \sum_{\alpha=1}^{N}\frac{\partial y_\alpha}{\partial x^i}\frac{\partial y_\alpha}{\partial x^j},$$

$$\sum_{\alpha=1}^{N}\big(g(\nabla y_\alpha, \nabla u)\big)^2 = |\nabla u|^2,$$
$$\sum_{\alpha=1}^{N} g(\nabla y_\alpha, \nabla y_\alpha) = \sum_{\alpha=1}^{N} |\nabla y_\alpha|^2 = n,$$
$$\sum_{\alpha=1}^{N} (\Delta y_\alpha)^2 = n^2|H|^2,$$
$$\sum_{\alpha=1}^{N} \Delta y_\alpha \nabla y_\alpha = 0, \tag{2.1}$$

where ∇ denotes the gradient operator on M^n and $|H|$ is the mean curvature of M^n.

Proof of Theorem 1.1. Let u_i be an eigenfunction corresponding to eigenvalue λ_i such that $\{u_i\}_{i=0}^{\infty}$ becomes an orthonormal basis of $L^2(M^n)$, that is,

$$\begin{cases} P_g u_i = \lambda_i u_i, \\ \int_{M^4} u_i u_j dv = \delta_{ij}, \quad i, j = 0, 1, \cdots. \end{cases}$$

We define an $N \times N$-matrix A as follows:

$$A := (a_{\alpha\beta})$$

where $a_{\alpha\beta} = \int_{M^4} y_\alpha u_0 u_\beta dv$, for $\alpha, \beta = 1, 2, \cdots, N$, and $\mathbf{y} = (y_\alpha)$ is the position vector of the immersion in $\mathbf{R}^N$. Using the orthogonalization of Gram and Schmidt, we know that there exist an upper triangle matrix $T = (T_{\alpha\beta})$ and an orthogonal matrix $U = (q_{\alpha\beta})$ such that $T = UA$, i.e.,

$$T_{\alpha\beta} = \sum_{\gamma=1}^{N} q_{\alpha\gamma} a_{\gamma\beta} = \int_{M^4} \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma u_0 u_\beta dv = 0, \quad 1 \le \beta < \alpha \le N.$$

Defining $z_\alpha = \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma$, we get

$$\int_{M^4} z_\alpha u_0 u_\beta dv = \int_{M^4} \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma u_0 u_\beta dv = 0, \quad 1 \le \beta < \alpha \le N.$$

Putting

$$\psi_\alpha := (z_\alpha - b_\alpha) u_0, \quad b_\alpha := \int_{M4} z_\alpha u_0^2 dv, \ 1 \le \alpha \le N,$$

we infer

$$\int_{M^4} \psi_\alpha u_\beta dv = 0, \quad 0 \le \beta < \alpha \le N.$$

Thus, from the Rayleigh–Ritz inequality, we have

$$\lambda_\alpha \int_{M^4} \psi_\alpha^2 dv \le \int_{M^4} \psi_\alpha P_g \psi_\alpha dv, \quad 1 \le \alpha \le N.$$

Since u_0 is a nonzero constant and

$$P_g \psi_\alpha = \Delta^2(z_\alpha u_0) - \operatorname{div}\left[\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)\nabla(z_\alpha u_0)\right], \tag{2.2}$$

according to the Stokes formula, we derive

$$\int_{M^4} \psi_\alpha P_g \psi_\alpha dv = \int_{M^4} \left[(\Delta z_\alpha)^2 u_0^2 + g\left(\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)\nabla z_\alpha, \nabla z_\alpha\right)u_0^2\right] dv.$$

From Lemma 2.1, we have

$$\begin{aligned}
&\sum_{\alpha=1}^{N} \int_{M^4} \psi_\alpha P_g \psi_\alpha dv \\
&\quad = \sum_{\alpha=1}^{N} \int_{M^4} \left[(\Delta z_\alpha)^2 u_0^2 + g\left(\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)\nabla z_\alpha, \nabla z_\alpha\right)u_0^2\right] dv \\
&\quad = \int_{M^4} \left(16|H|^2 + \frac{2}{3}R\right) u_0^2 dv.
\end{aligned}$$

Hence,

$$\sum_{\alpha=1}^{N} \lambda_\alpha \int_{M^4} \psi_\alpha^2 dv \le \int_{M^4} \left(16|H|^2 + \frac{2}{3}R\right) u_0^2 dv. \tag{2.3}$$

On the other hand,

$$\begin{aligned}
&\int_{M^4} \psi_\alpha (u_0 \Delta z_\alpha) dv \\
&\quad = \int_{M^4} (z_\alpha u_0 - u_0 b_\alpha)(u_0 \Delta z_\alpha) dv \\
&\quad = -\int_{M^4} \left|\nabla(z_\alpha u_0)\right|^2 dv.
\end{aligned}$$

Therefore, for any positive $\delta > 0$, we obtain from (2.3)

$$\begin{aligned}
&\lambda_\alpha^{\frac{1}{2}} \int_{M^4} \left|\nabla(z_\alpha u_0)\right|^2 dv \\
&\quad = -\lambda_\alpha^{\frac{1}{2}} \int_{M^4} \psi_\alpha (u_0 \Delta z_\alpha) dv \\
&\quad \le \frac{1}{2}\left(\delta\lambda_\alpha \int_{M^4} \psi_\alpha^2 dv + \frac{1}{\delta}\int_{M^4} (u_0\Delta z_\alpha)^2 dv\right) \\
&\sum_{\alpha=1}^N \lambda_\alpha^{\frac{1}{2}} \int_{M^4} \left|\nabla(z_\alpha u_0)\right|^2 dv \\
&\quad \le \frac{1}{2}\left(\delta \sum_{\alpha=1}^N \lambda_\alpha \int_{M^4} \psi_\alpha^2 dv + \frac{1}{\delta}\sum_{\alpha=1}^N \int_{M^4} (u_0\Delta z_\alpha)^2 dv\right) \\
&\quad \le \frac{1}{2}\left(\delta \int_{M^4} \left(16|H|^2 + \frac{2}{3}R\right) u_0^2 dv + \frac{1}{\delta}\int_{M^4} 16|H|^2 u_0^2 dv\right).
\end{aligned} \tag{2.4}$$

It is not hard to prove that, for any point and for any α,

$$|\nabla z_\alpha|^2 = g(\nabla z_\alpha, \nabla z_\alpha) \le 1.$$

Hence,

$$\begin{aligned}
&\sum_{\alpha=1}^N \lambda_\alpha^{\frac{1}{2}} |\nabla z_\alpha|^2 \\
&\quad \ge \sum_{i=1}^4 \lambda_i^{\frac{1}{2}} |\nabla z_i|^2 + \lambda_5^{\frac{1}{2}} \sum_{A=5}^N |\nabla z_A|^2 \\
&\quad = \sum_{i=1}^4 \lambda_i^{\frac{1}{2}} |\nabla z_i|^2 + \lambda_5^{\frac{1}{2}} \left(4 - \sum_{j=1}^4 |\nabla z_j|^2\right) \\
&\quad \ge \sum_{i=1}^4 \lambda_i^{\frac{1}{2}} |\nabla z_i|^2 + \sum_{j=1}^4 \lambda_j^{\frac{1}{2}} \left(1 - |\nabla z_j|^2\right) \\
&\quad \ge \sum_{j=1}^4 \lambda_j^{\frac{1}{2}}.
\end{aligned} \tag{2.5}$$

We obtain, by (2.4) and (2.5),

$$\int_M u_0^2 dv \sum_{j=1}^4 \lambda_j^{\frac{1}{2}} \le \frac{1}{2}\left(\delta \int_{M^4} \left(16|H|^2 + \frac{2}{3}R\right) u_0^2 dv + \frac{1}{\delta} \int_{M^4} 16|H|^2 u_0^2 dv\right).$$

Taking

$$\frac{1}{\delta} = \sqrt{\frac{\int_{M^4}(16|H|^2 + \frac{2}{3}R)u_0^2 dv}{\int_{M^4} 16|H|^2 u_0^2 dv}},$$

we have

$$\sum_{j=1}^4 \lambda_j^{\frac{1}{2}} \le 4\frac{\sqrt{\int_{M^4}(16|H|^2 + \frac{2}{3}R)dv \int_{M^4}|H|^2 dv}}{\mathrm{vol}(M^4)}. \tag{2.6}$$

If the equality holds, we have

$$\begin{aligned} \lambda_1 = \lambda_2 = \cdots = \lambda_N, \\ \Delta(z_\alpha - b_\alpha) = -\sqrt{\lambda_5}\delta(z_\alpha - b_\alpha). \end{aligned} \tag{2.7}$$

According to Takahashi's theorem, we know that M^4 is a round sphere $S^4(r)$ for $N = 5$ and M^4 is a minimal submanifold in a sphere $S^{N-1}(r)$ for $N > 5$ with $\sum_{\alpha=1}^N (z_\alpha - b_\alpha)^2 = r^2$. Thus, we have

$$\lambda_1 = \lambda_2 = \cdots = \lambda_N = \frac{16}{r^4\delta^2}.$$

From the definition of the Paneitz operator P_g, we have

$$P_g(z_\alpha - b_\alpha) = \Delta^2(z_\alpha - b_\alpha) - \mathrm{div}\left[\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)\nabla(z_\alpha - b_\alpha)\right], \tag{2.8}$$

that is, from (2.7) and (2.8), we have

$$\lambda_5\left(1 - \delta^2\right)(z_\alpha - b_\alpha) = -\mathrm{div}\left[\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)\nabla(z_\alpha - b_\alpha)\right].$$

According to $\sum_{\alpha=1}^N (z_\alpha - b_\alpha)^2 = r^2$, we obtain

$$\lambda_5\left(1 - \delta^2\right)r^2 = \sum_{\alpha=1}^N g\left(\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)\nabla(z_\alpha - b_\alpha), \nabla(z_\alpha - b_\alpha)\right).$$

Hence,

$$\lambda_5(1-\delta^2)r^2 = \frac{2}{3}R.$$

Thus, the scalar curvature R is constant. Hence, M^4 is a compact minimal submanifold with constant scalar curvature in a sphere $S^{N-1}(r)$. This finishes the proof of Theorem 1.1. □

Proof of Corollary 1.1. Since the unit sphere $S^N(1)$ is a hypersurface in $\mathbf{R}^{N+1}$ with the mean curvature 1, M^4 can be seen as a compact submanifold in $\mathbf{R}^{N+1}$ with the mean curvature $\sqrt{|H|^2+1}$. According to Theorem 1.1, we complete the proof of Corollary 1.1. □

3. Eigenvalues of the Paneitz operator on M^n ($n \neq 4$)

Proof of Theorem 1.2. Since $n > 4$, eigenvalues of the Paneitz operator P_g satisfy

$$\lambda_1 < \lambda_2 \leq \cdots, \lambda_k \leq \cdots \to +\infty.$$

Let u_i be an eigenfunction corresponding to eigenvalue λ_i such that $\{u_i\}_{i=1}^{\infty}$ becomes an orthonormal basis of $L^2(M^n)$, that is,

$$\begin{cases} P_g u_i = \lambda_i u_i, \\ \int_{M^n} u_i u_j dv = \delta_{ij}, \quad i, j = 1, 2, \cdots. \end{cases}$$

We shall use the same idea to prove Theorem 1.2. But, in this case, we need to use the first eigenfunction u_1, which is not constant in general. Thus, we need to compute many formulas. We define an $N \times N$-matrix A as follows:

$$A := (a_{\alpha\beta})$$

where $a_{\alpha\beta} = \int_{M^n} y_\alpha u_1 u_{\beta+1} dv$, for $\alpha, \beta = 1, 2, \cdots, N$, and $\mathbf{y} = (y_\alpha)$ is the position vector of the immersion in $\mathbf{R}^N$. Thus, there is an orthogonal matrix $U = (q_{\alpha\beta})$ such that

$$\int_{M^n} z_\alpha u_1 u_{\beta+1} dv = 0, \quad 1 \leq \beta < \alpha \leq N,$$

where $z_\alpha = \sum_{\gamma=1}^{N} q_{\alpha\gamma} y_\gamma$. Putting

$$\varphi_\alpha := (z_\alpha - a_\alpha)u_1, \quad a_\alpha := \int_{M^n} z_\alpha u_1^2 dv, \ 1 \leq \alpha \leq N,$$

we infer

$$\int_{M^n} \varphi_\alpha u_\beta dv = 0, \quad 1 \leq \beta \leq \alpha \leq N.$$

Thus, from the Rayleigh–Ritz inequality, we have

$$\lambda_{\alpha+1}\int_{M^n}\varphi_\alpha^2 dv \le \int_{M^n}\varphi_\alpha P_g\varphi_\alpha dv, \quad 1\le\alpha\le N. \tag{3.1}$$

$$\begin{aligned}
&P_g\varphi_\alpha = P_g(z_\alpha u_1) - a_\alpha P_g u_1 = P_g(z_\alpha u_1) - \lambda_1 a_\alpha u_1.\\
&P_g(z_\alpha u_1)\\
&\quad = \Delta^2(z_\alpha u_1) - \operatorname{div}\big[(a_n Rg + b_n \mathrm{Ric})\nabla(z_\alpha u_1)\big] + \frac{n-4}{2}Q(z_\alpha u_1)\\
&\quad = \Delta^2 z_\alpha\, u_1 + 2\Delta z_\alpha \Delta u_1 + 2\Delta g(\nabla z_\alpha, \nabla u_1)\\
&\qquad + 2g\big(\nabla z_\alpha, \nabla(\Delta u_1)\big) + z_\alpha\Delta^2 u_1 + 2g\big(\nabla(\Delta z_\alpha), \nabla u_1\big)\\
&\qquad - \operatorname{div}\big[u_1(a_n Rg + b_n\mathrm{Ric})\nabla z_\alpha\big] - \operatorname{div}\big[z_\alpha(a_n Rg + b_n\mathrm{Ric})\nabla u_1\big] + \frac{n-4}{2}Q(z_\alpha u_1)\\
&\quad = \Delta^2 z_\alpha u_1 + 2\Delta z_\alpha\Delta u_1 + 2\Delta g(\nabla z_\alpha, \nabla u_1) + 2g\big(\nabla z_\alpha, \nabla(\Delta u_1)\big) + 2g\big(\nabla(\Delta z_\alpha), \nabla u_1\big)\\
&\qquad - \operatorname{div}\big[u_1(a_n Rg + b_n\mathrm{Ric})\nabla z_\alpha\big] - g\big(\nabla z_\alpha, (a_n Rg + b_n\mathrm{Ric})\nabla u_1\big) + z_\alpha P u_1\\
&\quad = r_\alpha + \lambda_1 z_\alpha u_1
\end{aligned}$$

with

$$\begin{aligned}
r_\alpha = {}& \Delta^2 z_\alpha\, u_1 + 2\Delta z_\alpha\Delta u_1 + 2\Delta g(\nabla z_\alpha, \nabla u_1)\\
&+ 2g\big(\nabla z_\alpha, \nabla(\Delta u_1)\big) + 2g\big(\nabla(\Delta z_\alpha), \nabla u_1\big)\\
&- \operatorname{div}\big[u_1(a_n Rg + b_n\mathrm{Ric})\nabla z_\alpha\big] - g\big(\nabla z_\alpha, (a_n Rg + b_n\mathrm{Ric})\nabla u_1\big).
\end{aligned}$$

According to the Stokes formula, we derive

$$\int_{M^n} r_\alpha u_1 dv = 0.$$

Letting

$$\begin{aligned}
w_\alpha &= \int_{M^n} r_\alpha\varphi_\alpha dv\\
\int_{M^n}\varphi_\alpha P_g\varphi_\alpha dv &= \int_{M^n}\varphi_\alpha\big(P_g(z_\alpha u_1) - \lambda_1 a_\alpha u_1\big)dv\\
&= \int_{M^n}\varphi_\alpha(r_\alpha + \lambda_1\varphi_\alpha)dv.
\end{aligned}$$

Hence,

$$(\lambda_{\alpha+1}-\lambda_1)\int_{M^n}\varphi_\alpha^2 dv \le \int_{M^n}\varphi_\alpha r_\alpha dv = w_\alpha = \int_{M^n} z_\alpha u_1 r_\alpha dv, \quad 1\le\alpha\le N. \tag{3.2}$$

By a direct calculation, we obtain

$$\begin{aligned}
2\int_{M^n} z_\alpha u_1 g\big(\nabla(\Delta z_\alpha),\nabla u_1\big)dv &= \int_{M^n}(\Delta z_\alpha)^2 u_1^2 dv \\
&\quad + \int_{M^n}\Delta z_\alpha g\big(\nabla z_\alpha,\nabla u_1^2\big)dv - \int_{M^n}\big(z_\alpha\Delta^2 z_\alpha\big)u_1^2 dv,\\
2\int_{M^n} z_\alpha u_1 \Delta g(\nabla z_\alpha,\nabla u_1)dv &= 2\int_{M^n} u_1\Delta z_\alpha g(\nabla z_\alpha,\nabla u_1)dv \\
&\quad + 2\int_{M^n} z_\alpha\Delta u_1 g(\nabla z_\alpha,\nabla u_1)dv + 4\int_{M^n} g(\nabla z_\alpha,\nabla u_1)^2 dv\\
2\int_{M^n} z_\alpha u_1 g\big(\nabla z_\alpha,\nabla(\Delta u_1)\big)dv &= -2\int_{M^n} u_1 z_\alpha\Delta z_\alpha\Delta u_1 dv \\
&\quad - 2\int_{M^n} u_1\Delta u_1 g(\nabla z_\alpha,\nabla z_\alpha)dv - 2\int_{M^n} z_\alpha g(\nabla z_\alpha,\nabla u_1)\Delta u_1 dv.
\end{aligned}$$

Thus, we derive

$$\begin{aligned}
&(\lambda_{\alpha+1}-\lambda_1)\int_{M^n}\varphi_\alpha^2 dv\\
&\quad\le w_\alpha = \int_{M^n} z_\alpha u_1 r_\alpha dv = \int_{M^n}\big(u_1\Delta z_\alpha + 2g(\nabla z_\alpha,\nabla u_1)\big)^2 dv\\
&\qquad + \int_{M^n} u_1^2 g\big((a_n Rg + b_n \mathrm{Ric})\nabla z_\alpha,\nabla z_\alpha\big)dv - 2\int_{M^n} g(\nabla z_\alpha,\nabla z_\alpha)u_1\Delta u_1 dv,\\
&\quad 1\le\alpha\le N.
\end{aligned} \tag{3.3}$$

From Lemma 2.1, we have

$$\begin{aligned}
&\sum_{\alpha=1}^{N}(\lambda_{\alpha+1}-\lambda_1)\int_{M^n}\varphi_\alpha^2 dv\\
&\quad\le \int_{M^n}\big(n^2|H|^2 + (na_n+b_n)R\big)u_1^2 dv + 2(n+2)\int_{M^n} g(\nabla u_1,\nabla u_1)dv.
\end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned}
&\int_{M^n} \varphi_\alpha\big(u_1\Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1)\big)dv \\
&\quad = \int_{M^4} (z_\alpha - a_\alpha)u_1\big(u_1\Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1)\big)dv \\
&\quad = -\int_{M^4} |u_1\nabla z_\alpha|^2 dv. \qquad (3.5)
\end{aligned}$$

Therefore, for any positive $\delta > 0$, we obtain, from (3.5),

$$\begin{aligned}
&(\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}} \int_{M^n} |u_1\nabla z_\alpha|^2 dv \\
&\quad = -(\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}} \int_{M^n} \varphi_\alpha\big(u_1\Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1)\big)dv \\
&\quad \le \frac{1}{2}\left\{\delta(\lambda_{\alpha+1} - \lambda_1)\int_{M^n} \varphi_\alpha^2 dv + \frac{1}{\delta}\int_{M^n} \big(u_1\Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1)\big)^2 dv\right\}. \qquad (3.6)
\end{aligned}$$

According to (3.4) and (3.6), we infer

$$\begin{aligned}
&\sum_{\alpha=1}^{N}(\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}} \int_{M^n} |u_1\nabla z_\alpha|^2 dv \\
&\quad \le \frac{1}{2}\left\{\delta\sum_{\alpha=1}^{N}(\lambda_{\alpha+1} - \lambda_1)\int_{M^n} \varphi_\alpha^2 dv + \frac{1}{\delta}\sum_{\alpha=1}^{N}\int_{M^n} \big(u_1\Delta z_\alpha + 2g(\nabla z_\alpha, \nabla u_1)\big)^2 dv\right\} \\
&\quad \le \frac{1}{2}\delta\left\{\int_{M^n} \big(n^2|H|^2 + (na_n + b_n)R\big)u_1^2 dv + 2(n+2)\int_{M^n} g(\nabla u_1, \nabla u_1)dv\right\} \\
&\qquad + \frac{1}{2\delta}\left\{\int_{M^n} n^2|H|^2 u_1^2 dv + 4\int_{M^n} g(\nabla u_1, \nabla u_1)dv\right\}. \qquad (3.7)
\end{aligned}$$

By the same proof as the formula (2.5) in Section 2, we have

$$\sum_{\alpha=1}^{N}(\lambda_{\alpha+1} - \lambda_1)^{\frac{1}{2}}|\nabla z_\alpha|^2 \ge \sum_{j=1}^{n}(\lambda_{j+1} - \lambda_1)^{\frac{1}{2}}. \qquad (3.8)$$

Hence, we obtain

$$\begin{aligned}
&\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)^{\frac{1}{2}} \\
&\quad\le \frac{1}{2}\delta\Big(\int_{M^n}\big(n^2|H|^2+(na_n+b_n)R\big)u_1^2dv+2(n+2)\int_{M^n}g(\nabla u_1,\nabla u_1)dv\Big) \\
&\qquad+\frac{1}{2\delta}\Big(\int_{M^n}n^2|H|^2u_1^2dv+4\int_{M^n}g(\nabla u_1,\nabla u_1)dv\Big).
\end{aligned}\tag{3.9}$$

Letting S denote the squared norm of the second fundamental form of M^n, from the Gauss equation, we have

$$R=n(n-1)|H|^2-\big(S-n|H|^2\big)\le n(n-1)|H|^2.$$

Since

$$na_n+b_n=\frac{n^2-2n-4}{2(n-1)}>0,$$

we have

$$n^2|H|^2+(na_n+b_n)R\le\frac{n(n^2-4)|H|^2}{2}.$$

Taking

$$\frac{1}{\delta}=\sqrt{\frac{\int_{M^n}n^2|H|^2u_1^2dv+4\int_{M^n}g(\nabla u_1,\nabla u_1)dv}{\int_{M^n}\frac{n(n^2-4)|H|^2}{2}u_1^2dv+2(n+2)\int_{M^n}g(\nabla u_1,\nabla u_1)dv}}$$

we have

$$\begin{aligned}
&\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)^{\frac{1}{2}} \\
&\quad\le\sqrt{\int_{M^n}\frac{n(n^2-4)|H|^2}{2}u_1^2dv+2(n+2)\int_{M^n}g(\nabla u_1,\nabla u_1)dv} \\
&\qquad\times\sqrt{\int_{M^n}n^2|H|^2u_1^2dv+4\int_{M^n}g(\nabla u_1,\nabla u_1)dv}.
\end{aligned}\tag{3.10}$$

If the equality holds, we have

$$\lambda_2=\lambda_3=\cdots=\lambda_N,$$

and $S \equiv n|H|^2$. Thus, M^n is totally umbilical, that is, M^n is isometric to a sphere. It completes the proof of Theorem 1.2. □

Corollary 3.1. *Let (M^n, g) $(n > 4)$ be an n-dimensional compact submanifold with the metric g in the unit sphere $S^N(1)$. Then, eigenvalues of the Paneitz operator P_g satisfy*

$$\begin{aligned}
&\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)^{\frac{1}{2}} \\
&\quad \le \sqrt{\int_{M^n} \frac{n(n^2-4)(|H|^2+1)}{2} u_1^2 dv + 2(n+2)\int_{M^n} g(\nabla u_1, \nabla u_1) dv} \\
&\qquad \times \sqrt{\int_{M^n} n^2\big(|H|^2+1\big)u_1^2 dv + 4\int_{M^n} g(\nabla u_1, \nabla u_1) dv}
\end{aligned} \tag{3.11}$$

and the equality holds if and only if M^n is isometric to a sphere $S^n(r)$, where u_1 is the normalized first eigenfunction of P_g.

Proof of Theorem 1.3. Since $n \neq 4$, we assume that eigenvalues of the Paneitz operator P_g satisfy

$$0 < \lambda_1 < \lambda_2 \le \cdots, \lambda_k \le \cdots \to +\infty.$$

Let u_i be an eigenfunction corresponding to eigenvalue λ_i such that $\{u_i\}_{i=1}^{\infty}$ becomes an orthonormal basis of $L^2(M^n)$, that is,

$$\begin{cases} P_g u_i = \lambda_i u_i, \\ \int_{M^n} u_i u_j dv = \delta_{ij}, \quad i, j = 1, 2, \cdots. \end{cases}$$

We shall use the similar method to prove Theorem 1.3. We define an $(N+1)\times(N+1)$-matrix A as follows:

$$A := (a_{\alpha\beta})$$

where $a_{\alpha\beta} = \int_{M^n} y_\alpha u_\beta dv$, for $\alpha, \beta = 1, 2, \cdots, N+1$, and $\mathbf{y} = (y_\alpha)$ is the position vector of the immersion in $\mathbf{R}^{N+1}$ with $|\mathbf{y}|^2 = \sum_{\alpha=1}^{N+1} y_\alpha^2 = 1$. Thus, there is an orthogonal matrix $U = (q_{\alpha\beta})$ such that

$$\int_{M^n} z_\alpha u_\beta dv = 0, \quad 1 \le \beta < \alpha \le N+1,$$

where $z_\alpha = \sum_{\gamma=1}^{N+1} q_{\alpha\gamma} y_\gamma$. Since U is an orthogonal matrix, we have

$$\sum_{\alpha=1}^{N+1} z_\alpha^2 = 1.$$

Putting

$$\psi_\alpha := z_\alpha, \quad 1 \le \alpha \le N+1,$$

we infer

$$\int_{M^n} \psi_\alpha u_\beta dv = 0, \quad 1 \le \beta < \alpha \le N+1.$$

Thus, from the Rayleigh–Ritz inequality, we have

$$\lambda_\alpha \int_{M^n} \psi_\alpha^2 dv \le \int_{M^n} \psi_\alpha P_g \psi_\alpha dv, \quad 1 \le \alpha \le N+1,$$
$$P_g \psi_\alpha = P_g(z_\alpha). \tag{3.12}$$

According to the Stokes formula, we derive

$$\int_{M^n} \psi_\alpha P_g \psi_\alpha dv = \int_{M^n} \left[(\Delta z_\alpha)^2 + g\big((a_n Rg + b_n \mathrm{Ric})\nabla z_\alpha, \nabla z_\alpha\big) + \frac{n-4}{2} Q(z_\alpha)^2 \right] dv$$

From Lemma 2.1, we have

$$\begin{aligned}
&\sum_{\alpha=1}^{N+1} \int_{M^n} \psi_\alpha P_g \psi_\alpha dv \\
&= \sum_{\alpha=1}^{N+1} \int_{M^n} \left[(\Delta z_\alpha)^2 + g\big((a_n Rg + b_n \mathrm{Ric})\nabla z_\alpha, \nabla z_\alpha\big) + \frac{n-4}{2} Q(z_\alpha)^2 \right] dv \\
&= \int_{M^n} \left(\big(n^2|H|^2 + n^2\big) + (na_n + b_n)R + \frac{n-4}{2} Q \right) dv.
\end{aligned} \tag{3.13}$$

Hence,

$$\sum_{\alpha=1}^{N+1} \lambda_\alpha \int_{M^n} \psi_\alpha^2 dv \le \int_{M^n} \left(\big(n^2|H|^2 + n^2\big) + (na_n + b_n)R + \frac{n-4}{2} Q \right) dv. \tag{3.14}$$

On the other hand,

$$\int_{M^n} \psi_\alpha (\Delta z_\alpha) dv = \int_{M^n} z_\alpha \Delta z_\alpha dv = -\int_{M^n} |\nabla z_\alpha|^2 dv. \tag{3.15}$$

Therefore, for any positive $\delta > 0$, we obtain

$$\begin{aligned}
&\lambda_\alpha^{\frac{1}{2}} \int_{M^n} |\nabla z_\alpha|^2 dv \\
&= -\lambda_\alpha^{\frac{1}{2}} \int_{M^n} \psi_\alpha (\Delta z_\alpha) dv \\
&\le \frac{1}{2}\left(\delta \lambda_\alpha \int_{M^n} \psi_\alpha^2 dv + \frac{1}{\delta} \int_{M^n} (\Delta z_\alpha)^2 dv\right)
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\sum_{\alpha=1}^{N+1} \lambda_\alpha^{\frac{1}{2}} \int_{M^n} |\nabla z_\alpha|^2 dv \\
&\le \frac{1}{2}\left(\delta \sum_{\alpha=1}^{N+1} \lambda_\alpha \int_{M^n} \psi_\alpha^2 dv + \frac{1}{\delta} \sum_{\alpha=1}^{N+1} \int_{M^n} (\Delta z_\alpha)^2 dv\right) \\
&\le \frac{1}{2}\left[\delta \int_{M^n} \left(\left(n^2|H|^2 + n^2\right) + (na_n + b_n)R + \frac{n-4}{2} Q\right) dv \right. \\
&\quad \left. + \frac{1}{\delta} \int_{M^n} \left(n^2|H|^2 + n^2\right) dv\right].
\end{aligned} \tag{3.17}$$

By using the same proof as the formula (2.5) in Section 2, we have

$$\sum_{\alpha=1}^{N+1} \lambda_\alpha^{\frac{1}{2}} |\nabla z_\alpha|^2 \ge \sum_{j=1}^{n} \lambda_j^{\frac{1}{2}}. \tag{3.18}$$

Thus, we obtain

$$\begin{aligned}
\sum_{j=1}^{n} \lambda_j^{\frac{1}{2}} \operatorname{vol}\left(M^n\right) \le \frac{1}{2}\Bigg[&\delta \int_{M^n} \left(\left(n^2|H|^2 + n^2\right) + (na_n + b_n)R + \frac{n-4}{2} Q\right) dv \\
&+ \frac{1}{\delta} \int_{M^n} \left(n^2|H|^2 + n^2\right) dv\Bigg].
\end{aligned} \tag{3.19}$$

Taking

$$\frac{1}{\delta} = \sqrt{\frac{\int_{M^n} ((n^2|H|^2 + n^2) + (na_n + b_n)R + \frac{n-4}{2} Q) dv}{\int_{M^n} (n^2|H|^2 + n^2) dv}}$$

we have

$$\sum_{j=1}^{n} \lambda_j^{\frac{1}{2}} \le n \frac{\sqrt{\int_{M^n}((n^2|H|^2+n^2)+(na_n+b_n)R+\frac{n-4}{2}Q)dv \int_{M^n}(|H|^2+1)dv}}{\mathrm{vol}(M^n)}.$$

If the equality holds, we have

$$\lambda_2 = \lambda_3 = \cdots = \lambda_{N+1},$$

$|\nabla z_1| \equiv 1$ because of $\lambda_1 < \lambda_2$ and

$$\Delta z_1 = -\sqrt{\lambda_1}\delta z_1, \qquad \Delta z_\alpha = -\sqrt{\lambda_n}\delta z_\alpha \quad \text{for } \alpha > 1.$$

Since

$$\sum_{\alpha=1}^{N+1} z_\alpha^2 = 1,$$

we have

$$n - \sqrt{\lambda_n}\delta + (\sqrt{\lambda_n}\delta - \sqrt{\lambda_1}\delta)z_1^2 = 0.$$

Thus,

$$\sqrt{\lambda_n}\delta = \sqrt{\lambda_1}\delta$$

or z_1^2 is constant. It is impossible because $|\nabla z_1| \equiv 1$ and $\lambda_1 < \lambda_2$. Therefore, the equality does not hold. It completes the proof of Theorem 1.3. □

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